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Calculation of generalized Lommel integrals for modified Bessel functions

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Abstract. Results are presented for both definite and indefinite integrals of certain products of two modified Bessel functions K_ν. General recurrence relations are developed for these integrals which depend on both the order of the modified Bessel functions and various parameters. Explicit low-order formulae and special cases are given and many of these have application to mathematical and physical problems where the Green function for the Helmholtz operator in two dimensions (K_0) appears.

1. Introduction

In this paper we will be concerned with the calculation of integrals of certain products of modified Bessel functions. We present results for integrals of the form

$$I_{p,q,\gamma} \equiv \int x^\gamma K_p(\alpha x) K_q(\beta x)\,\mathrm{d}x \tag{1}$$

where K_ν is a modified Bessel function of the second kind of order ν. Here α, β and γ are real numbers and because $K_\nu = K_{-\nu}$ [13] (p. 970) we restrict attention to $p, q \geqslant 0$. We refer to $I_{p,q,\gamma}$ as a generalized Lommel integral in relation to the usual Lommel integrals which are for products of Bessel functions J_ν on a finite interval with $\gamma = 0$ or 1 [14, 21]. Although there is an abundance of published results for integrals of products of Bessel functions J_ν (e.g., [9, 17, 21, 22, 28]) there does not appear to be a comprehensive collection of results for indefinite integrals of the form (1). The relative scarcity of integral forms (1) in such well known works as [1, 10, 13, 18, 23] is to be noted. It is hoped that the formulae presented here, especially the low-order cases tabulated in section 3, will partially fill this gap and provide a useful reference.

We would particularly like to mention the application of our results for low orders p, q in (1) in phenomenological theories of superfluidity and type II superconductivity. The zero-order modified Bessel function K_0 is especially important in certain physical applications where it provides a Green function for the Helmholtz operator $\nabla^2 - 1/\lambda^2$ in two dimensions in cylindrical coordinates (e.g., [20]). In the London theory of superconductivity the function

$$b(r) = \frac{\phi_0}{2\pi\lambda^2} K_0(r/\lambda)$$

where λ, ϕ_0 are physical constants (λ being the penetration depth and ϕ_0 the flux quantum), represents the magnetic field due to an azimuthal supercurrent [2, 4, 5, 11, 27]. That is, $b(r) \propto K_0(r/\lambda)$ models a magnetic vortex of a single flux quantum. Because of this connection, integrals of products of modified Bessel functions appear in the calculation of many physical quantities. For example, various energy densities are proportional to the square of b and integrals and derivatives of b. In turn, from the line energy of a magnetic vortex the lower critical field H_{c1} may be calculated [4, 7, 25, 27]. Another example of the occurrence of integrals of the form (1) is provided by the calculation of the viscous drag on a moving vortex, as in the continuum version of the model in [6]. The integrals (1) for γ a negative integer seem to arise most frequently in these physical problems.

In this paper we present results for both indefinite and definite integrals. The formulas for evaluation of definite integrals should be useful for numerical purposes. In section 2 we present a simple two-term recurrence relation (RR) for $I_{p,q,\gamma}$ and its solution for general p, q, and γ a negative integer. We also discuss the need to write the RRs in terms of decreasing orders p, q. In section 3 we specialize a RR of section 2 to integer order cases for negative integer values of γ for small $|\gamma|$. The resulting indefinite integrals are collected there. In sections 4 and 5 we present other RRs for $I_{p,q,\gamma}$ which are three-term in one of the parameters p, q, or γ. (The RRs of sections 2 and 4 are made in terms of decreasing orders.) Special cases of the RRs of sections 4 and 5 also provide useful results for indefinite integrals. The results of section 4 are derived from the defining differential equation for modified Bessel functions while the method of section 5 uses a RR for the derivative of K_ν and a suitable factoring of x^γ. The general RRs which are developed may be amenable to implementation in computer algebra systems such as MACSYMA, Mathematica, REDUCE, Scratchpad, or SMP [15, 24, 26, 30, 31]. The paper concludes with a brief summary.

We note that commercial software exists which will symbolically integrate elementary functions and numerically integrate a range of special functions (e.g., [15, 24, 26, 30, 31]). As some of these software packages have the means to recognize and manipulate certain special functions, including modified Bessel functions, it would appear possible to incorporate, say, the formulae of section 3, into such software to provide an automated capability to symbolically integrate products of modified Bessel functions. Furthermore, insofar as existing packages contain integration rules, such as integration by parts, it seems possible that software tools could be used to generate RRs such as we present and iterate them to any finite number of terms. These are only a few of the possibilities for software implementation.

2. Two-term recursion relation for $I_{p,q,\gamma}$

Before presenting the major results of this section, we remark on a few of the possibilities for evaluating integrals of products of modified Bessel functions. The use of a generating function is often a useful technique for calculating integrals of special functions. However, because of the presence of the logarithmic singularity in K_ν, this option, at least in the usual power series form, is not available. Among the remaining possibilities of evaluating integrals of the form (1) are manipulation of the series or integral representations of K_ν. In the former case integration term-by-term and in the second case interchange of the order of integration, followed by re-expression in terms of K_ν will be required. In view of the tedium of these approaches we have opted for the development of various recurrence relations (RRs) for $I_{p,q,\gamma}$. Our methods appear to be complementary to the method of [12] which uses an equivalent system of differential equations.

We now derive the first and perhaps simplest of our RRs, which still yields many practical results. We use the two RRs for derivatives of K_ν [13]

$$\frac{\mathrm{d}K_\nu(\alpha x)}{\mathrm{d}x} = -\frac{\nu}{x}K_\nu(\alpha x) - \alpha K_{\nu-1}(\alpha x) \tag{2}$$

$$\frac{\mathrm{d}K_\nu(\alpha x)}{\mathrm{d}x} = \frac{\nu}{x}K_\nu(\alpha x) - \alpha K_{\nu+1}(\alpha x). \tag{3}$$

Define

$$I_{p,q,\gamma}(\alpha, \beta, a, b) \equiv \int_a^b x^\gamma K_p(\alpha x)K_q(\beta x)\,\mathrm{d}x \tag{4}$$

and

$$\begin{aligned} M_{p,q,\gamma}&(\alpha, \beta, a, b) \\ &\equiv [x^\gamma K_p(\alpha x)K_q(\beta x)]_a^b \\ &\equiv b^\gamma K_p(\alpha b)K_q(\beta b) - a^\gamma K_p(\alpha a)K_q(\beta a). \end{aligned} \tag{5}$$

Two obvious symmetries of $M_{p,q,\gamma}$ are

$$M_{p,q,\gamma}(\alpha, \beta, a, b) = M_{q,p,\gamma}(\beta, \alpha, a, b) = -M_{p,q,\gamma}(\alpha, \beta, b, a).$$

We also note that because K_ν vanishes at infinity $-M_{p,q,\gamma}(\alpha, \beta, z, \infty) = z^\gamma K_p(\alpha z)K_q(\beta z)$, thereby providing a connection between definite and indefinite integrals.

By integration by parts and (2) we have

$$I_{p,q,\gamma} = \frac{-1}{(p+q-\gamma-1)}[\alpha I_{p-1,q,\gamma+1} + \beta I_{p,q-1,\gamma+1} + M_{p,q,\gamma+1}] \qquad p+q-\gamma \neq 1 \tag{6}$$

and by (3) we have

$$I_{p,q,\gamma} = \frac{1}{(p+q+\gamma+1)}[\alpha I_{p+1,q,\gamma+1} + \beta I_{p,q+1,\gamma+1} + M_{p,q,\gamma+1}] \qquad p+q+\gamma \neq -1. \tag{7}$$

We refer to (6) and (7) as two-term RRs for $I_{p,q,\gamma}$ because they are two-term in each index p and q. In figures 1(a) and 1(b) we give the 'stencil' for RRs (6) and (7). These

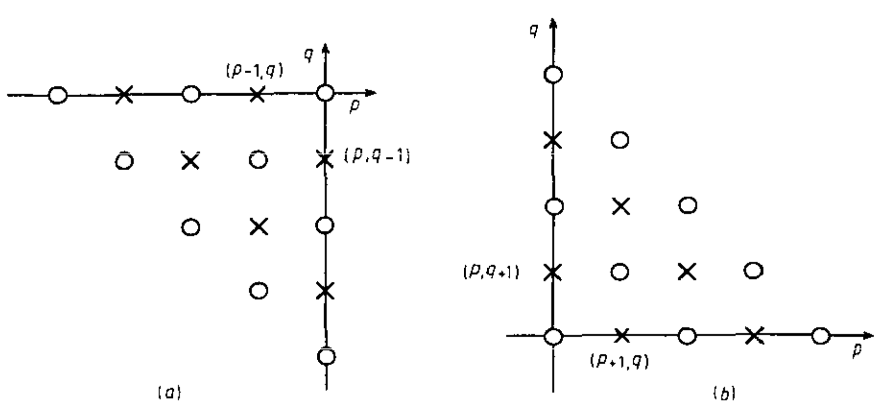

Figure 1. (a) Stencil for RR (6) in the pq plane. The alternating cross and circle symbols denote successive generations. (b) Stencil for RR (7) in the pq plane. As for (a), the nth generation contains $n+1$ distinct points.

figures are akin to the computational 'molecules' employed in describing finite difference schemes (e.g., [16], p 45). In figure 1(a), for instance, the point at coordinates (p, q) represents integral $I_{p,q,\gamma}$. By (6), it leads in the 'first generation' to neighbouring points $(p-1, q)$ and $(p, q-1)$. Implicit in figure 1 is that we move from a sheet with index γ to a sheet with index $\gamma+1$ at each generation. It is easy to see in figure 1(a) that the nth generation has $n+1$ points, representing $n+1$ integrals, lying on the line $p+q=-n$ in pq space. Similarly, for the nth generation of RR (7) in figure 1(b), $n+1$ points lie on the line $p+q=n$. By counting each time that a given integral contributes to the next generation, figures 1(a) and 1(b) also yield the number of times each integral $I_{p,q,\gamma}$ appears in the nth generation. Indeed, for RR (6), by appending a '1' to the point (p, q), a Pascal triangle is formed with each row of the triangle corresponding to the line $p+q=-n$. Similarly, the powers of α and β at each point may be included in such figures.

The RR (6) appears quite generally useful, for it expresses an integral $I_{p,q,\gamma}$ in terms of integrals of lower order. Here we will focus on the case that γ is a negative integer: by iteration we can reduce $I_{p,q,\gamma}$ to a sum of $M_{p,q,\gamma}$ terms and integrals with $\gamma=-1$. In fact, for γ a negative integer with $|\gamma|\geqslant 2$, we may write a general formula, setting $\delta\equiv|\gamma+1|=-\gamma-1$. Then we have

$$I_{p,q,\gamma}=\prod_{\mu=0}^{\delta-1}\frac{-1}{(p+q+\delta-2\mu)}\sum_{k=0}^{\delta}\binom{\delta}{k}\alpha^{\delta-k}\beta^{k}I_{p-\delta+k,q-k,-1}$$
$$+\prod_{\mu=0}^{l}\frac{-1}{(p+q+\delta-2\mu)}\sum_{l=0}^{\delta-1}\sum_{k=0}^{l}\binom{l}{k}\alpha^{l-k}\beta^{k}M_{p-l+k,q-k,l-\delta}. \tag{8}$$

For integer orders, the integrals $I_{p-1,p,0}$, $I_{p,p,-1}$ and $I_{p,q,-1}$ can be done directly with the aid of the RR

$$K_{\nu+1}(x)=K_{\nu-1}(x)+\frac{2\nu}{x}K_{\nu}(x) \tag{9}$$

and are given in section 3 for $\alpha=\beta=1$. The integral $I_{p,q,-1}$ may be expressed in terms of $M_{p,q,\gamma}$ and inserted into (8) to obtain a formula for $I_{p,q,\gamma}$ in terms of $M_{p,q,\gamma}$ only. This result will not be given here as it is not needed further on. Instead, the results of the reduction process for indefinite integrals for γ a negative integer and integer orders are collected in section 3.

The RR (7) can be iterated for general γ, resulting in a series of $M_{p,q,\gamma}$ terms with increasing order. It is then possible to write $I_{p,q,\gamma}$ as a formal infinite series:

$$I_{p,q,\gamma}=\sum_{m=0}^{\infty}\sum_{n=0}^{m}D_{m,n}(p,q,\alpha,\beta,\gamma)M_{p+m,q+n,\gamma+n+1} \tag{10}$$

where

$$D_{m,n}(p,q,\alpha,\beta,\gamma)=\frac{\binom{m}{n}\alpha^{m-n}\beta^{n}}{\prod_{\mu=0}^{m+1}(p+q+\gamma+1+2\mu)}\qquad n\leqslant m. \tag{11}$$

The simplicity of formula (10), (11) is due to the fact that (7) is only a two-term RR. However, (10) is valid only when the infinite series converges. To test the convergence or divergence of the series we form certain asymptotic estimates [8] which are not presented here. We find that the series in fact diverges due to the rapid growth of modified Bessel functions in the order. Therefore RRs such as (7), which are developments in increasing order, are not useful when taken to an infinite number of terms.

3. Indefinite integrals with γ a negative integer

In this section we give results of the use of RR (6) for γ a negative integer and take $\alpha=\beta=1$. The integrals (12)–(14) may be performed directly by using RR (9). We note, however, that certain $\gamma=0$, 1 integrals must be left in terms of $\int K_0^2\,dx$. This latter integral is only expressible as an infinite sum [17].

For m even,

$$\int K_m K_{m+1}\,dx = -\tfrac{1}{2}K_0^2 + K_1^2 - K_2^2 + \ldots + K_{m-1}^2 - K_m^2 \qquad m \geqslant 0. \tag{12a}$$

For n odd,

$$\int K_n K_{n+1}\,dx = \tfrac{1}{2}K_0^2 - K_1^2 + K_2^2 - \ldots + K_{n-1}^2 - K_n^2. \tag{12b}$$

For m even,

$$\int \frac{1}{x} K_m^2\,dx = -\frac{1}{2m}(K_0^2 - 2K_1^2 + 2K_2^2 - \ldots - 2K_{m-1}^2 + K_m^2) \qquad m \neq 0. \tag{13a}$$

For n odd,

$$\int \frac{1}{x} K_n^2\,dx = -\frac{1}{2n}(-K_0^2 + 2K_1^2 - 2K_2^2 + \ldots - 2K_{n-1}^2 + K_n^2). \tag{13b}$$

$$\begin{aligned}\int \frac{1}{x} K_n K_m\,dx &= -\frac{1}{n+m} K_n K_m - \frac{x}{(m^2-n^2)}[K_n K_{m-1} - K_{n-1} K_m] \\ &= \frac{1}{n+m} K_n K_m - \frac{x}{(m^2-n^2)}[K_n K_{m+1} - K_{n+1} K_m] \qquad m \neq n\end{aligned} \tag{14}$$

$$\int \frac{1}{x^2} K_n^2\,dx = -\frac{1}{(2n+1)}\frac{K_n^2}{x} + \frac{2}{(4n^2-1)} K_{n-1}K_n - \frac{2x}{(4n^2-1)}(K_n K_{n-2} - K_{n-1}^2) \tag{15}$$

$$\begin{aligned}\int \frac{1}{x^2} K_{n-1} K_n\,dx &= \frac{1}{2}\frac{1}{n(n-1)}[-K_0^2 + 2K_1^2 - 2K_2^2 + \ldots - 2K_{n-2}^2 + (2-n)K_{n-1}^2 + (n-1)K_n^2] \\ &\quad + \frac{1}{x} K_n K_{n-1} \qquad n \neq 0, 1\end{aligned} \tag{16}$$

$$\begin{aligned}\int \frac{1}{x^2} K_n K_m\,dx &= -\frac{1}{(n+m+1)}\frac{K_n K_m}{x} + \frac{1}{(n+m)^2-1}(K_{n-1}K_m + K_n K_{m-1}) \\ &\quad - \frac{2x}{[(n+m)^2-1][(n-m)^2-1]} K_{n-1}K_{m-1} - \frac{x}{(n+m+1)} \\ &\quad \times \left[\frac{1}{[m^2-(n-1)^2]} K_{n-2}K_m - \frac{1}{[(m-1)^2-n^2]} K_n K_{m-2}\right] \qquad |n-m| \neq 1\end{aligned} \tag{17}$$

$$\int \frac{1}{x^3} K_n^2 \, \mathrm{d}x = -\frac{1}{2(n+1)} \frac{K_n^2}{x^2} - \frac{1}{2(n+1)} \left\{ \frac{1}{2} \frac{1}{n(n-1)} [-K_0^2 + 2K_1^2 - 2K_2^2 + \ldots - 2K_{n-2}^2 \right.$$

$$\left. + (2-n)K_{n-1}^2 + (n-1)K_n^2] + \frac{1}{x} K_n K_{n-1} \right\} \qquad n \neq 0, 1. \tag{18}$$

4. Application of ODE

Here we give an alternative method of developing RRs for $I_{p,q,\gamma}$, which are in general three-term in γ. This approach is based on partial integration of the defining ordinary differential equation (ODE) for modified Bessel functions. The ODE for $K_p(\alpha x)$ may be written in the form

$$K_p(\alpha x) = \frac{1}{\alpha^2} \left[\frac{\mathrm{d}^2 K_p(\alpha x)}{\mathrm{d}x^2} + \frac{1}{x} \frac{\mathrm{d}K_p(\alpha x)}{\mathrm{d}x} - \frac{p^2}{x^2} K_p(\alpha x) \right]. \tag{19}$$

If we multiply equation (19) by $x^\gamma K_q(\beta x)$ and integrate from a to b we have

$$I_{p,q,\gamma} = \frac{1}{\alpha^2} \left[\int_a^b x^\gamma K_q(\beta x) \frac{\mathrm{d}^2 K_p(\alpha x)}{\mathrm{d}x^2} \, \mathrm{d}x + \int_a^b x^{\gamma-1} K_q(\beta x) \frac{\mathrm{d}K_p(\alpha x)}{\mathrm{d}x} \, \mathrm{d}x \right.$$

$$\left. - p^2 \int_a^b x^{\gamma-2} K_q(\beta x) K_p(\alpha x) \, \mathrm{d}x \right]. \tag{20}$$

We define

$$N_{p,q,\gamma}(\alpha, \beta, a, b) \equiv \left[x^\gamma \left(K_q(\beta x) \frac{\mathrm{d}K_p(\alpha x)}{\mathrm{d}x} - K_p(\alpha x) \frac{\mathrm{d}K_q(\beta x)}{\mathrm{d}x} \right) \right]_a^b \tag{21}$$

(which can be related to $M_{p,q,\gamma}$ by RRs (2), (3)). We integrate the first term on the right-hand side of equation (20) by parts twice and use the ODE (19) for $K_q(\beta x)$ in the form

$$\frac{\mathrm{d}^2 K_q(\beta x)}{\mathrm{d}x^2} + \frac{1}{x} \frac{\mathrm{d}K_q(\beta x)}{\mathrm{d}x} = \left(\frac{q^2}{x^2} + \beta^2 \right) K_q(\beta x).$$

These operations result in

$$I_{p,q,\gamma} = \frac{1}{\alpha^2} \left[N_{p,q,\gamma} + (1-\gamma) \int_a^b x^{\gamma-1} K_q(\beta x) \frac{\mathrm{d}K_p(\alpha x)}{\mathrm{d}x} \, \mathrm{d}x \right.$$

$$+ \int_a^b \left(\frac{q^2}{x^2} + \beta^2 \right) x^\gamma K_q(\beta x) K_p(\alpha x)$$

$$\left. + (\gamma-1) \int_a^b x^{\gamma-1} K_p(\alpha x) \frac{\mathrm{d}K_q(\beta x)}{\mathrm{d}x} \, \mathrm{d}x - p^2 I_{p,q,\gamma-2} \right]$$

$$= \frac{1}{\alpha^2} \left\{ N_{p,q,\gamma} + (1-\gamma) \int_a^b x^{\gamma-1} \left[K_q(\beta x) \frac{\mathrm{d}K_p(\alpha x)}{\mathrm{d}x} - K_p(\alpha x) \frac{\mathrm{d}K_q(\beta x)}{\mathrm{d}x} \right] \mathrm{d}x \right.$$

$$\left. + \beta^2 I_{p,q,\gamma} + (q^2 - p^2) I_{p,q,\gamma-2} \right\}. \tag{22}$$

We now employ RR (2) with the result

$$I_{p,q,\gamma} = \frac{1}{(\alpha^2-\beta^2)}\{N_{p,q,\gamma} + (q^2-p^2)I_{p,q,\gamma-2} + (1-\gamma)(q-p)I_{p,q,\gamma-2}$$

$$+(1-\gamma)[-\alpha I_{p-1,q,\gamma-1} + \beta I_{p,q-1,\gamma-1}]\} \qquad \alpha \neq \beta. \tag{23}$$

The RR (23) is two-term in the orders p, q but three-term in γ. Figure 2 presents a 'stencil' for (23) in the (p, γ) plane. As for figure 1, a number can be attached to each point in figure 2 giving the number of times the corresponding integral appears in that generation. A diagram in the (q, γ) plane would be similar. It is seen that in the nth generation in figure 2 points lie on the $n+1$ 'diagonal' lines given by $p+\gamma=-n, -n-1, \ldots, -2n$. The corresponding number of distinct integrals in the nth generation is $\frac{1}{2}(n+1)(n+2)$.

The RR (23) may be expanded, reducing $I_{p,q,\gamma}$ to a sum of terms of lower order. This RR seems most useful for γ a positive integer and some special cases. We may enumerate these cases as follows.

(1) For $\gamma=1$ we obtain

$$I_{p,q,1} = \frac{1}{(\alpha^2-\beta^2)}[(q^2-p^2)I_{p,q,-1} + N_{p,q,1}] \qquad \alpha \neq \beta \tag{24}$$

a standard Lommel integral [21].

(2) For $\alpha=\beta$ we have

$$I_{p,q,\gamma} = \frac{1}{(q-p)(q+p-\gamma-1)}[-N_{p,q,\gamma+2}(\alpha,\alpha) + (\gamma+1)\alpha(-I_{p-1,q,\gamma+1} + I_{p,q-1,\gamma+1})]$$

$$p \neq q \qquad q+p \neq \gamma+1 \tag{25}$$

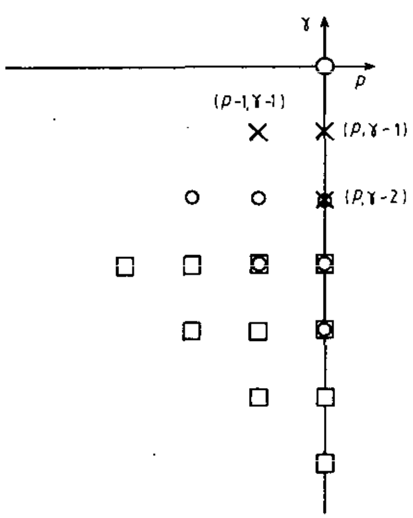

Figure 2. The first three generations for RR (23) in the $p\gamma$ plane are shown. The crosses denote the first generation, the circles mark the second generation, and the squares the third generation.

which further reduces for $\gamma=-1$ to

$$I_{p,q,-1}=-\frac{1}{(q^2-p^2)}N_{p,q,1}(\alpha,\alpha). \tag{26}$$

In view of RRs *(2), (3), this is seen to agree with formula (14) of section 3.*

(3) For $p=q$,

$$I_{p,p,\gamma}=\frac{1}{(\alpha^2-\beta^2)}\{N_{p,p,\gamma}+(1-\gamma)[-\alpha I_{p-1,\gamma-1}+\beta I_{p,p-1,\gamma-1}]\}. \tag{27}$$

That is, the RR (23) becomes two-term in p, γ. For $\gamma=1$, (27) gives the Lommel integral

$$I_{p,p,1}=\frac{1}{(\alpha^2-\beta^2)}N_{p,p,1}. \tag{28}$$

5. Recursion for $I_{p,q,-\gamma}$ constant in $-\gamma$

In this section we develop RRs for $I_{p,q,-\gamma}$ which do not vary the parameter $-\gamma$. This approach is based on the relation [13]

$$x^{-q+1}K_q(\xi x)=-\frac{1}{\xi}\frac{\mathrm{d}}{\mathrm{d}x}(x^{-q+1}K_{q-1}(\xi x)) \tag{29}$$

integration by parts, and a suitable factoring of $x^{-\gamma}$.

By a simple change of variable $X=\alpha x$, with the definitions $\xi\equiv\beta/\alpha$, $a'\equiv\alpha a$, $b'\equiv\alpha b$, we have $I_{p,q,-\gamma}=I'_{p,q}/\alpha^{-\gamma+1}$ where

$$I'_{p,q}\equiv\int_{a'}^{b'}X^{-\gamma}K_p(X)K_q(\xi X)\,\mathrm{d}X. \tag{30}$$

We now write $X^{-q+1}X^{q-\gamma-1}$ for the $X^{-\gamma}$ factor in (30) and use (29). Then integration by parts yields

$$I'_{p,q}=A_{p,q}-\frac{1}{\xi}\alpha^{-\gamma}M_{p,q-1,-\gamma}(\alpha,\beta,a,b)$$

where

$$A_{p,q}\equiv\frac{1}{\xi}\int_{a'}^{b'}X^{-q+1}K_{q-1}(\xi X)[(q-\gamma-1)X^{q-\gamma-2}K_p(X)+X^{q-\gamma-1}K'_p(X)]\,\mathrm{d}X \tag{31}$$

and $M_{p,q,-\gamma}$ is defined in (5). Using RR (2) for K'_p we have

$$A_{p,q}=\frac{1}{\xi}\int_{a'}^{b'}[(q-p-\gamma-1)X^{-\gamma-1}K_p(X)K_{q-1}(\xi X)$$
$$-X^{-\gamma}K_{p-1}(X)K_{q-1}(\xi X)]\,\mathrm{d}X. \tag{32}$$

The second integral on the right-hand side of (32) is just $-(1/\xi)I'_{p-1,q-1}$. Next we employ the RR

$$K_{q-1}(\xi X)=\frac{\xi X}{2(q-1)}[K_q(\xi X)-K_{q-2}(\xi X)] \qquad q\neq 1 \tag{33}$$

so that $A_{p,q}$ becomes

$$A_{p,q}=\frac{(q-p-\gamma-1)}{2(q-1)}\int_{a'}^{b'}[X^{-\gamma}K_p(X)K_q(\xi X)-X^{-\gamma}K_p(X)K_{q-2}(\xi X)]\,\mathrm{d}X$$

$$-\frac{1}{\xi}I'_{p-1,q-1}$$

$$=\frac{(q-p-\gamma-1)}{2(q-1)}[I'_{p,q}-I'_{p,q-2}]-\frac{1}{\xi}I'_{p-1,q-1}\qquad q\neq 1. \tag{34}$$

Therefore we obtain the result

$$I_{p,q,-\gamma}=\frac{(p-q+\gamma+1)}{(p+q+\gamma-1)}I_{p,q-2,-\gamma}-\frac{2(q-1)}{\xi(p+q+\gamma-1)}I_{p-1,q-1,-\gamma}$$

$$-\frac{2(q-1)}{\beta(p+q+\gamma-1)}M_{p,q-1,-\gamma}\qquad p+q+\gamma\neq 1. \tag{35}$$

If we had instead used the factorization $X^{-\gamma}=X^{-p+1}X^{p-\gamma-1}$ in (30) we would have obtained the RR

$$I_{p,q,-\gamma}=\frac{(q-p+\gamma+1)}{(p+q+\gamma-1)}I_{p-2,q,-\gamma}-\frac{2\xi(p-1)}{(p+q+\gamma-1)}I_{p-1,q-1,-\gamma}$$

$$-\frac{2(p-1)}{\alpha(p+q+\gamma-1)}M_{p-1,q,-\gamma}\qquad p+q+\gamma\neq 1. \tag{36}$$

The RR (35) is two-term in the order p and three-term in the order q while RR (36) is three-term in p and two-term in q. Both of these RRs are constant in the parameter

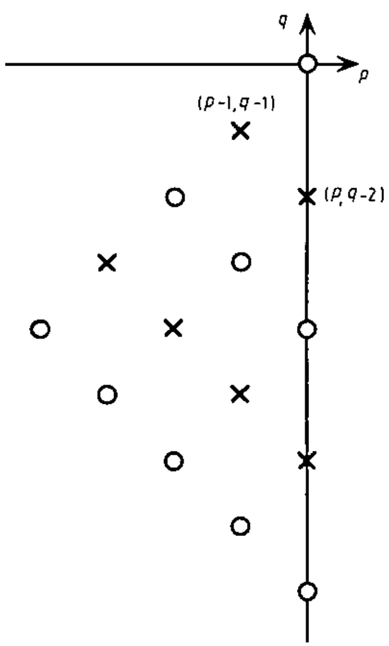

Figure 3. Stencil for RR (35) in the pq plane. Successive generations of integrals are marked by cross or circle symbols.

$-\gamma$. They have been derived in such a way that $I_{p,q,-\gamma}$ is expressed in terms of integrals of lower order, as we saw in section 2 that RRs developed in terms of higher order have limited usefulness. For the special cases that $q=1$ in (35) or $p=1$ in (36), these RRs reduce to the identity $I_{1,q,-\gamma}=I_{-1,q,-\gamma}$. For the special case that $p+q+\gamma-1=0$, the RRs (35), (36) become, respectively,

$$I_{p,q-2,-\gamma}+\frac{1}{\xi}I_{p-1,q-1,-\gamma}=-\frac{1}{\beta}M_{p,q-1,-\gamma} \qquad p-1=-\gamma-q \tag{37a}$$

$$I_{p-2,q,-\gamma}+\xi I_{p-1,q-1,-\gamma}=-\frac{1}{\alpha}M_{p-1,q,-\gamma} \qquad q-1=-\gamma-p. \tag{37b}$$

The RRs (37) are increasing in one of the orders while decreasing in the other. The RR (37a) appears useful in the case that the orders are integral and $p<0$ and $q>0$, while (37b) appears best suited for $p>0$, $q<0$.

Supposing that p and q are positive integers, the RRs (35) or (36) may be used to reduce an integral $I_{p,q,-\gamma}$ to one of the form $I_{\nu,0,-\gamma}$, i.e., of the form

$$I_{\nu,0,-\gamma}=\int_a^b x^{-\gamma}K_0(\beta x)K_\nu(\alpha x)\,\mathrm{d}x. \tag{38}$$

By repeated application of RR (9) for successive K_ν, we may further reduce (38) to the evaluation of integrals $I_{0,0,-\mu}$ and $I_{0,1,-\mu'}$.

Finally, referring to figure 3, we see that the nth generation of RR (35) (or (36)) has $n+1$ integrals, the corresponding points lying on the line $p+q=-2n$.

6. Summary

In this paper we pointed out the numerous occurrences of integrals of the form (1) in physical applications. We developed and discussed general recursion relations (RRs) for the integrals $I_{p,q,\gamma}$ where p, q are the orders of the modified Bessel functions and γ is the power of a factor of the integration variable. We presented a geometrical aid, in the stencils of figures 1–3, for the analysis of the RRs. We mentioned that the RRs for the integrals (1) appear to be amenable to implementation in computer algebra systems and, in fact, such programs in the near future may be able to serve in deriving these or similar relations.

The various RRs that we derived have complementary ranges of usefulness. The simple two-term RR (6), derived in section 2 by integration by parts and the basic RRs (2), (3) for modified Bessel functions, seems well suited for the parameter γ a negative integer. A collection of low-order results from RR (6) appears in section 3. In sections 4 and 5 we presented RRs which are three-term in one of the parameters p, q or γ. The results of section 4 were derived from the defining ordinary differential equation for modified Bessel functions while the results of section 5 came from the use of a RR for the derivative of K_ν and a factoring of x^γ. The RR (23) of section 4 is well suited for γ a positive integer. The special cases (24)–(28) of RR (23) hold for $\gamma=\pm1$, $\alpha=\beta$ or $p=q$. Some of these special cases reduce to well known simple Lommel integrals. The RRs (35), (36) of section 5 should prove useful when both the orders are positive integers, while the special cases (37) should be useful when one order is negative and the other positive.

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